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Stabilisation of Linear Reduced Order Model of Aerofoil Gust Response via Restarting

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A method has been proposed for stabilising a linear Reduced Order Model of aerofoil gust response. The reduced model is based on linear subspace system identification in the frequency domain. It is shown that by applying shifts to remove the unstable eigenvalues from the reduced system, a stable reduced order model can be generated. A further method is suggested for correcting the steady state response of the reduced model to match exactly with the full order steady state response. The created reduced system is tested by prescribing a series of 1-cosine gusts and comparing the output to the full order simulations.

Nomenclature

A_r, B_r, C_r, D_r	= reduced state space matrices	$M + 1$	= number of frequency response samples
\bar{B}_r, \bar{D}_r	= corrected B_r and D_r matrices	μ	= unstable reduced system eigenvalue
$G(z)$	= system transfer function	ω_k	= discrete gust excitation frequency
\hat{H}	= block Hankel matrix	\mathbf{u}	= discrete input vector
j	= imaginary unit	\mathbf{x}	= state vector
\mathcal{L}	= Lagrangian function	\mathbf{y}	= output vector

I. Introduction

THE motivation for developing Reduced Order Models (ROM) comes from the necessity to reduce the amount of time required to run Computational Fluid Dynamics (CFD) simulations of aircraft fluid-structure interactions. Full order simulations are often prohibitively time intensive and processor hungry, and therefore require large scale computational resources to complete. It would therefore be desirable to create models of full order systems which exhibit the same dynamic characteristics, but take a fraction of the time to run ¹.

Reduced order modelling has been an area of active research in the aerospace community. Amongst the many methods of ROM generation, the two most prevalent methods employed in the literature are Proper Orthogonal Decomposition (POD) and its variants ²⁻⁴, and Eigenvalue Realisation Algorithm (ERA) ^{5,6}. Both methods rely on obtaining datasets of the full order system and involve projection onto a set of modes ⁵.

Particular research has also been focused on ROM generation of aerofoil gust response ⁷⁻⁹. CFD problems involving gusts are usually highly dynamic and require simulations over large time periods, giving an additional motivation for the development of ROMs.

The research carried out in the current study is a continuation of previous research completed by Bagheri et al ¹⁰, where a method was developed to create a frequency domain based ROM of a 2D rigid aerofoil subject to external gust excitations. The full simulation of the gust was developed using the Split Velocity Method (SVM) ¹¹, accounting for the mutual interaction between the gust and the aerofoil. This study further develops and enhances the same ROM by ensuring ROM stability using restarting, and by correcting the steady state response of the ROM to match exactly with the full order model. The new ROMs have been tested by prescribing a series of 1-cosine gusts, and comparing the ROM outputs to the results from the full order simulations, where very good agreement between the sets of results is observed.

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II. Background

A. System Identification and Model Reduction

A reduced model of a system represented as a discrete Linear Time-Invariant (LTI), Multiple-Input & Multiple-Output (MIMO) system, written in state-space form, is given by:

$$\begin{aligned} \mathbf{x}(n+1) &= A_r \mathbf{x}(n) + B_r \mathbf{u}(n) \\ \mathbf{y}(n) &= C_r \mathbf{x}(n) + D_r \mathbf{u}(n) \end{aligned} \quad (1)$$

where A_r, B_r, C_r and D_r are the reduced discrete system matrices.

The linear system identification algorithm given by McKelvey et al.¹² forms the basis of the model reduction method incorporated in this work, and is briefly discussed in this section. The identification algorithm is based on obtaining the frequency response of the system and using the Inverse Discrete Fourier Transform (IDFT) to obtain what is in effect the system impulse response. Starting from $M+1$ equispaced discrete frequencies between 0 and π , the frequency response G is obtained by running frequency domain simulations of the system with harmonic inputs at the discrete frequencies:

$$\omega_k = \frac{\pi k}{M} \quad k = 0, \dots, M \quad (2)$$

The frequency response is then extended to the full unit circle by taking the complex conjugate:

$$G_{M+k+1} = G_{M-k+1}^* \quad k = 1, \dots, M-1 \quad (3)$$

A block Hankel matrix \hat{H} is then defined as follows, comprising of the IDFT of the transfer function:

$$\hat{H} \stackrel{\text{def}}{=} \begin{bmatrix} \hat{h}_1 & \hat{h}_2 & \dots & \hat{h}_M \\ \hat{h}_2 & \hat{h}_3 & \dots & \hat{h}_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{h}_M & \hat{h}_{M+1} & \dots & \hat{h}_{2M-1} \end{bmatrix} \in \mathbb{R}^{M \times M} \quad (4)$$

$$\hat{h}_i \stackrel{\text{def}}{=} \frac{1}{2M} \sum_{k=0}^{2M-1} G_k e^{\frac{j2\pi i k}{2M}} \quad i = 0, \dots, 2M-1 \quad (5)$$

A Singular Value Decomposition (SVD) of the Hankel matrix is taken and truncated to retain the r largest singular values:

$$\hat{H} = [\hat{U}_r \quad \hat{U}_o] \begin{bmatrix} \hat{\Sigma}_r & 0 \\ 0 & \hat{\Sigma}_o \end{bmatrix} \begin{bmatrix} \hat{V}_r^T \\ \hat{V}_o^T \end{bmatrix} \quad (6)$$

The reduced system matrices are then determined as:

$$\begin{aligned} A_r &= (J_1 \hat{U}_r)^\dagger J_2 \hat{U}_r \\ C_r &= J_3 \hat{U}_r \end{aligned} \quad (7)$$

where

$$\begin{aligned} J_1 &= [I_{M-1} \quad 0]_{M-1 \times M} \\ J_2 &= [0 \quad I_{M-1}]_{M-1 \times M} \\ J_3 &= [I \quad 0_{M-1}]_{1 \times M} \end{aligned} \quad (8)$$

and $X^\dagger = (X^T X)^{-1} X^T$ is the Moore-Penrose pseudoinverse of the full column rank matrix X .

Lastly, B_r and D_r are found by solving a least squares problem. Initially, the following new matrices are defined:

$$\hat{\chi} := \begin{bmatrix} C_r(z_0 I - A_r) & I \\ C_r(z_1 I - A_r) & I \\ \vdots & \vdots \\ C_r(z_M I - A_r) & I \end{bmatrix}, \quad \hat{\mathcal{G}} := \begin{bmatrix} G_0 \\ G_1 \\ \vdots \\ G_M \end{bmatrix} \quad (9)$$

where

$$z_k = e^{j\omega_k} \quad (10)$$

for the range of excitation frequencies. The least squares estimate is then given by:

$$\begin{bmatrix} B_r \\ D_r \end{bmatrix} = \begin{bmatrix} \text{Re } \hat{\chi} \\ \text{Im } \hat{\chi} \end{bmatrix}^\dagger \begin{bmatrix} \text{Re } \hat{\mathcal{G}} \\ \text{Im } \hat{\mathcal{G}} \end{bmatrix} \quad (11)$$

which gives the estimated transfer function as:

$$\tilde{G}(z) = D_r + C_r(zI - A_r)^{-1} B_r \quad (12)$$

This reduction algorithm was used to find the ROM of a 2-D rigid aerofoil subject to gust excitations in transonic flight. As discussed, the physical interaction between the aerofoil and the oncoming gust was modelled using the Split Velocity Method ¹¹. Computational simulations of the system were performed by solving the linearised inviscid Euler equations in the frequency domain to obtain the system frequency response. For a more detailed discussion of the SVM equations and the model reduction refer to Bagheri et al. ¹⁰

B. Restarting

For a reduced system to be stable, the eigenvalues of A_r have to remain inside the unit disk. The motivation for system restarting is to remove any unstable eigenvalues from the reduced system. This is achieved by applying shifts to remove the undesirable eigenvalues and to identify a new system which will not have the same stability issues ¹³. The shift is applied to the reduced C_r matrix, and a new Hankel matrix is formed. The subsequent components of the system identification algorithm to create a new reduced system remain unchanged. Restarting can be performed successively until a stable reduced system has been found.

Starting from Eq. (5), and writing the frequency response G as the Fourier Transform of the systems impulse response g_l ¹² we have:

$$G_l = \sum_{l=0}^{\infty} g_l e^{-j\omega l} = \sum_{l=0}^{\infty} g_l e^{\frac{\pi k l}{M}} \quad (13)$$

$$\begin{aligned} \therefore \hat{h}_i &= \frac{1}{2M} \sum_{k=0}^{2M-1} \sum_{l=0}^{\infty} g_l e^{\frac{j2\pi k(i-l)}{2M}} = \sum_{l=0}^{\infty} g_l + 2lM = C A^{i-1} \left(\sum_{l=0}^{\infty} A^{2lM} \right) B \\ &= C A^{i-1} (I - A^{2M})^{-1} B \end{aligned} \quad (14)$$

Note that the equation above also holds true for the reduced system, and therefore the reduced system matrices can be substituted in the relationship.

To restart an unstable reduced system, the following transformation is applied to the reduced C_r matrix:

$$\bar{C} = C_r(A_r - \mu_1 I) \quad (15)$$

Where the shift, μ_1 , is the undesired eigenvalue of A_r .

Substituting the above equation into Eq. (14), we have:

$$\begin{aligned}
\hat{h}_i &= C_r(A_r - \mu_1 I)A_r^{i-1}(I - A_r^{2M})^{-1}B_r \\
&= C_r A_r^i (I - A_r^{2M})^{-1}B_r - \mu_1 C_r A_r^{i-1}(I - A_r^{2M})^{-1}B_r \\
&= \hat{h}_{i+1} - \mu_1 \hat{h}_i
\end{aligned} \tag{16}$$

Therefore the new Hankel matrix becomes:

$$\hat{H} = \hat{H}_{i+1} - \mu_1 \hat{H}_i \tag{17}$$

To remove a complex pair of unstable eigenvalues, a complex pair of shifts is applied to transform C_r

$$\bar{C} = C_r(A_r - \mu_2 I)(A_r - \mu_1 I) \tag{18}$$

Which gives the following new Hankel matrix:

$$\hat{H} = \hat{H}_{i+2} - (\mu_1 + \mu_2)\hat{H}_{i+1} + \mu_1 \mu_2 \hat{H}_i \tag{19}$$

It should be noted that in order to construct \hat{H}_{i+1} or, if required, \hat{H}_{i+2} , new IDFT parameters will be needed to fill in the Hankel matrix. However since the number of data points is fixed at the start of the model reduction process, there will be a fixed number of IDFT elements. Therefore, each time restarting is applied the number of data points taken for the system reduction has to be reduced by one or two, depending on how many eigenvalues are being removed. This would ensure that the Hankel matrix can be constructed with the correct elements. After restarting has been carried out, the reduced C_r matrix is obtained by the following transformation:

$$C_r = \bar{C} \left(\prod_{i=1}^n (A_r - \mu_i I)^{-1} \right) \tag{20}$$

Where n is the number of shifts that have been applied.

C. Steady State Correction

The steady state response of the reduced system can be corrected to ensure the response at zero frequency matches exactly with the response of the full order system. From Eq. (11) the following expression can be written for the estimated reduced system frequency response:

$$\hat{\chi} \begin{bmatrix} B_r \\ D_r \end{bmatrix} = \tilde{G}(z) \tag{21}$$

The product of the first row of the $\hat{\chi}$ matrix with $\begin{bmatrix} B_r \\ D_r \end{bmatrix}$ gives the first entry of the estimated frequency response, which is the steady state gain, or the response at zero frequency. Therefore, to ensure the steady state response of the estimated reduced system matches with the full order system, either of the following methods could be employed:

1. Adjust D_r

By simply shifting the D_r matrix the entire reduced frequency response will be shifted so that the steady state gain matches that of the full order system. This however is the least rigorous solution.

$$\begin{aligned}
\hat{\chi}_{1:} \begin{bmatrix} B_r \\ D_r \end{bmatrix} &= \tilde{G}_1 \\
\therefore \bar{D}_r &= \tilde{G}_1 - C_r(I - A_r)^{-1}B_r
\end{aligned} \tag{22}$$

2. Adjust B_r

This adjustment can be made by finding the least amount of change that needs to be applied to the B_r matrix elements to ensure steady state matching. It can be done by changing B_r by B_δ and minimising B_δ by using the Moore-Penrose pseudoinverse.

$$\begin{aligned}\hat{\chi}_{1:} \begin{bmatrix} B_r + B_\delta \\ D_r \end{bmatrix} &= \tilde{G}_1 \\ \therefore [C_r(z_0 I - A_r)] B_\delta &= \tilde{G}_1 - D_r - [C_r(z_0 I - A_r)] B_r \\ B_\delta &= [C_r(z_0 I - A_r)]^\dagger (\tilde{G}_1 - D_r - [C_r(z_0 I - A_r)] B_r) \\ \bar{B}_r &= B_r + B_\delta\end{aligned}\tag{23}$$

3. Constrained Least Squares

The least squares problem given by Eq. (11) can be solved with an added constrain to ensure the steady state response of the reduced system matches that of the full order system. The following equation is solved:

$$\text{Minimise } \left\| \tilde{G}(z) - \hat{\chi} \begin{bmatrix} B_r \\ D_r \end{bmatrix} \right\|^2 \text{ such that } \hat{\chi}_{1:} \begin{bmatrix} B_r \\ D_r \end{bmatrix} = \tilde{G}_1\tag{24}$$

The problem is solved by forming the following Lagrangian Function, with Lagrange multiplier γ

$$\mathcal{L}([B_r, D_r]^T, \gamma) = \left\| \tilde{G}(z) - \hat{\chi} \begin{bmatrix} B_r \\ D_r \end{bmatrix} \right\|^2 + \gamma \left(\hat{\chi}_{1:} \begin{bmatrix} B_r \\ D_r \end{bmatrix} - \tilde{G}_1 \right)\tag{25}$$

The optimality conditions are:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial [B_r, D_r]^T}([B_r, D_r]^T, \gamma) &= 2(\hat{\chi}^\dagger \hat{\chi}) \begin{bmatrix} B_r \\ D_r \end{bmatrix} - 2\hat{\chi}^T \tilde{G}(z) + \gamma \hat{\chi}_{1:} = 0 \\ \frac{\partial \mathcal{L}}{\partial \gamma}([B_r, D_r]^T, \gamma) &= \hat{\chi}_{1:} \begin{bmatrix} B_r \\ D_r \end{bmatrix} - \tilde{G}_1 = 0\end{aligned}\tag{26}$$

Putting the above relations together in an augmented matrix gives the Karush-Kuhn-Tucker (KKT) ¹⁴ conditions:

$$\begin{bmatrix} 2(\hat{\chi}^\dagger \hat{\chi}) & \hat{\chi}_{1:}^T \\ \hat{\chi}_{1:} & 0 \end{bmatrix} \begin{bmatrix} B_r \\ D_r \end{bmatrix} = \begin{bmatrix} 2\hat{\chi}^T \tilde{G}(z) \\ \tilde{G}_1 \end{bmatrix}\tag{27}$$

Which, assuming that the KKT matrix is invertible, gives the new \bar{B}_r & \bar{D}_r matrices as below:

$$\begin{bmatrix} \bar{B}_r \\ \bar{D}_r \end{bmatrix} = \begin{bmatrix} 2(\hat{\chi}^\dagger \hat{\chi}) & \hat{\chi}_{1:}^T \\ \hat{\chi}_{1:} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2\hat{\chi}^T \tilde{G}(z) \\ \tilde{G}_1 \end{bmatrix}\tag{28}$$

The KKT matrix is invertible if and only if

$$\hat{\chi}_{1:} \text{ has independent rows, and } \begin{bmatrix} \hat{\chi} \\ \hat{\chi}_{1:} \end{bmatrix} \text{ has independent columns}$$

Which is true for the ROM. This is the most accurate method for steady state correction, and it the method used in this work.

III. Results

As discussed, a ROM has been generated for a 2D rigid aerofoil in transonic flow under external excitations due to gusts. The results from two aerofoil and flow configurations are presented here. Results have been included to draw comparisons between cases where restarting and steady state correction have and have not been applied. The test cases are given below.

Table 1. Test cases for different aerofoil & Mach number configurations

Case	Aerofoil	Mach Number
1	NACA0010	0.735
2	NACA2410	0.25

Euler C-grid meshes were created around the aerofoils with a grid sizing of 181×60 , with 100 cells on the aerofoil surface. As an example the NACA0010 grid is given in the figure below.

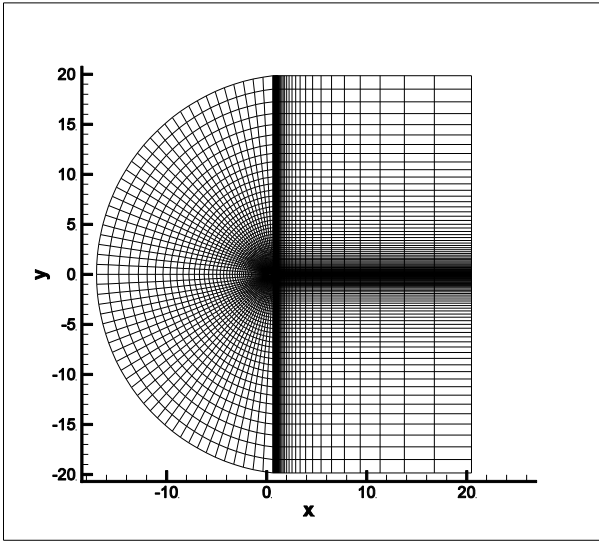


Figure 1. C-grid around NACA0010 aerofoil

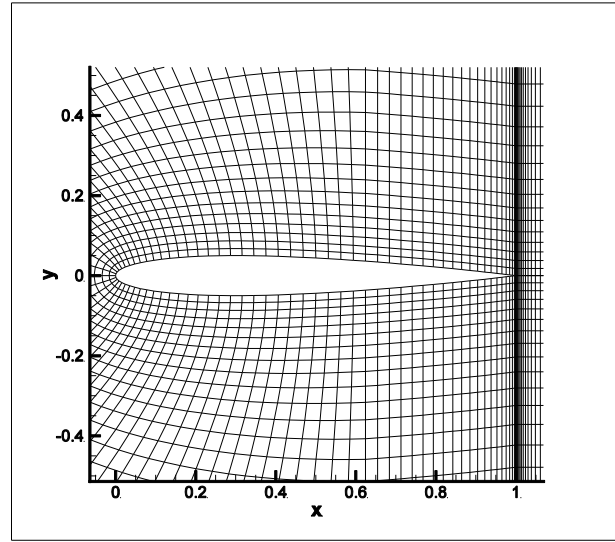


Figure 2. NACA0010 aerofoil surface grid

The steady pressure distribution for the test cases listed in Table 1 can be found below:

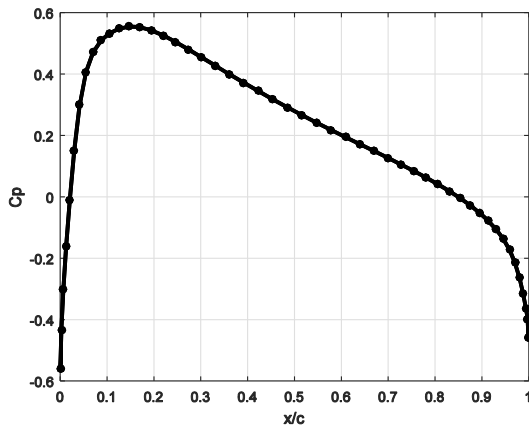


Figure 3. Steady pressure distribution for case 1

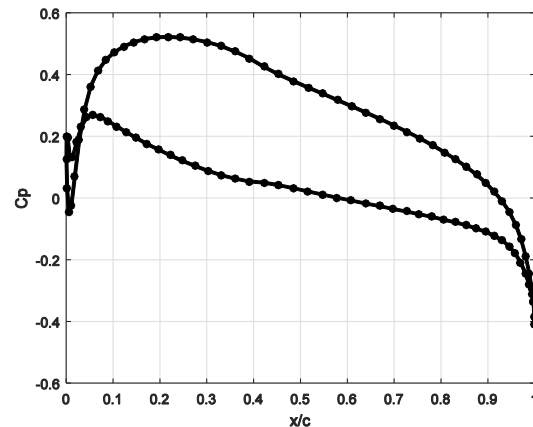


Figure 4. Steady pressure distribution for case 2

The figures below illustrate the generated ROMs for both test cases. The magnitude and phase of the frequency response of the original and reduced system are given. In the cases below, no steady state correction or restarting has been applied and the ROMs found are unstable. The discrete eigenvalues are also given, showing that both systems are unstable without restarting. In both cases, there are a total of 257 data points for the full system, and the size of the reduced system is 25.

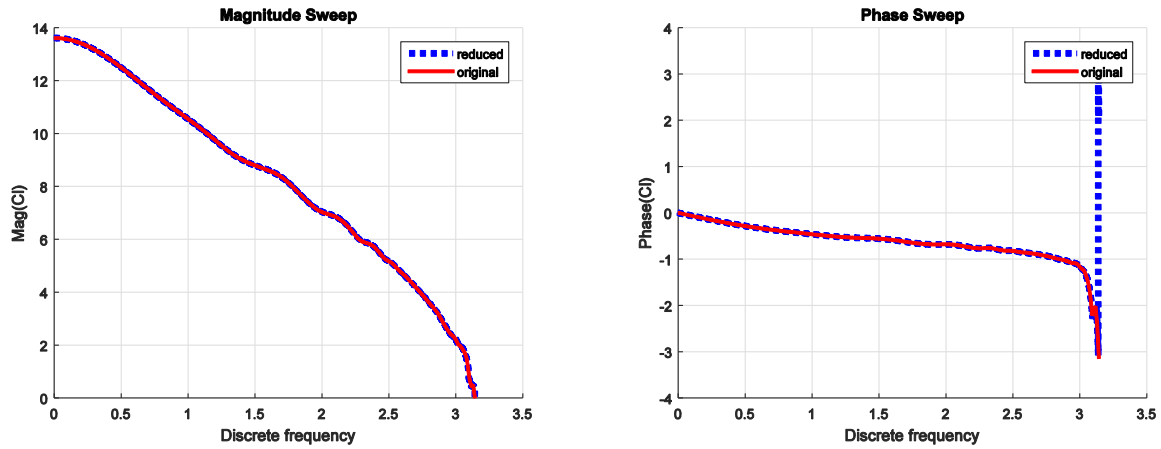


Figure 5. Magnitude & phase plot of ROM for case 1

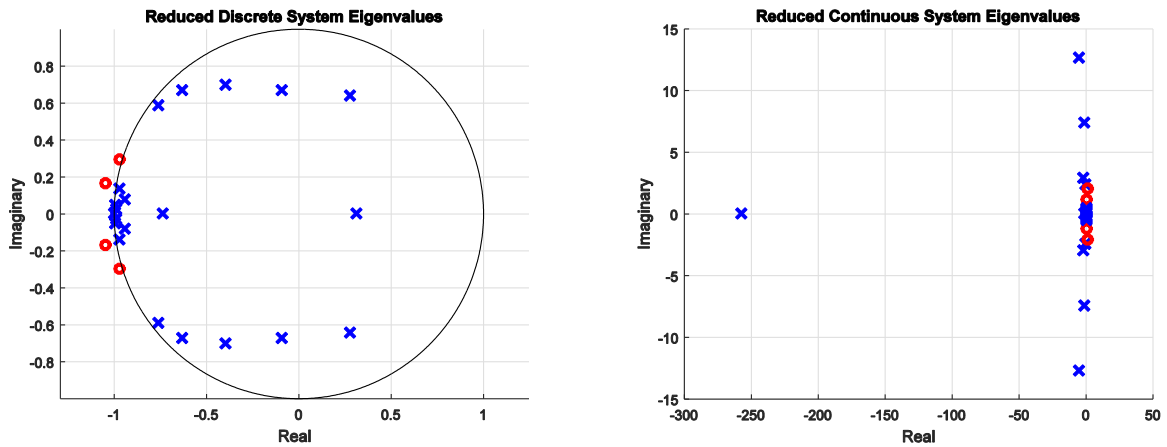


Figure 6. Eigenvalues of unstable ROM for case 1

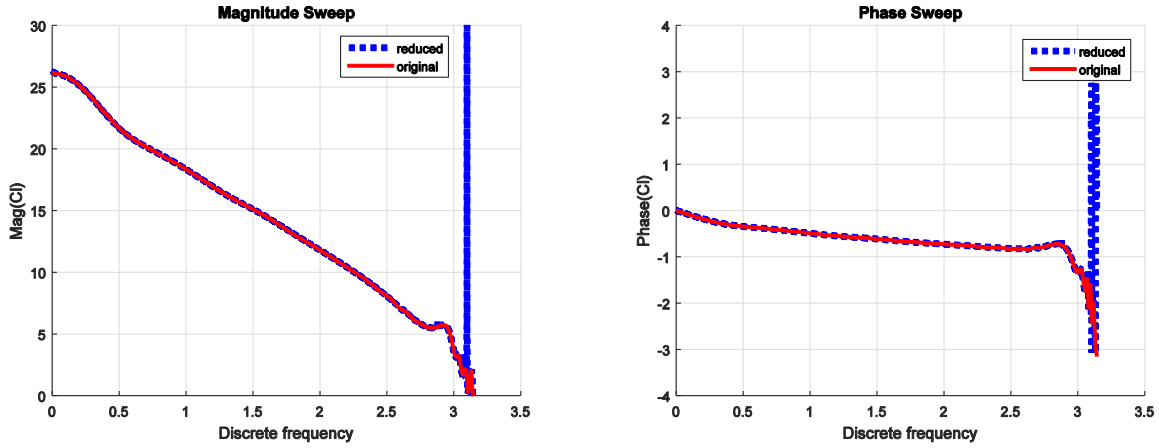


Figure 7. Magnitude & phase plot of ROM for case 2

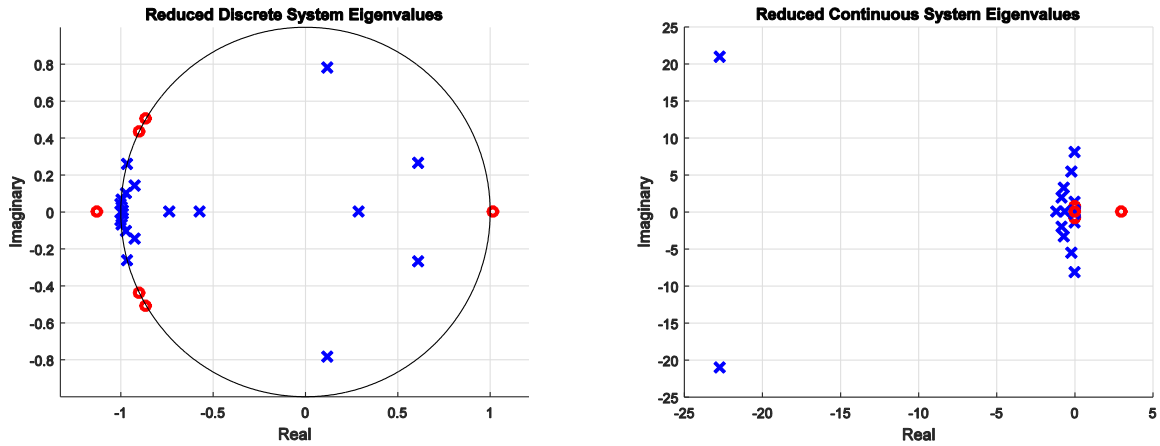


Figure 8. Eigenvalues of unstable ROM for case 2

Both systems become stable after applying restarting. It was found that for case 1, restarting had to be applied 3 times in succession to obtain a stable system, while for case 2 it had to be applied 7 times. The stable ROMs obtained are shown in the figures below, where it can be seen that neither system has any unstable eigenvalues remaining after restarting. Although not readily apparent in the plots below, steady state correction has also been applied by creating the KKT matrices and solving for the new reduced matrices.

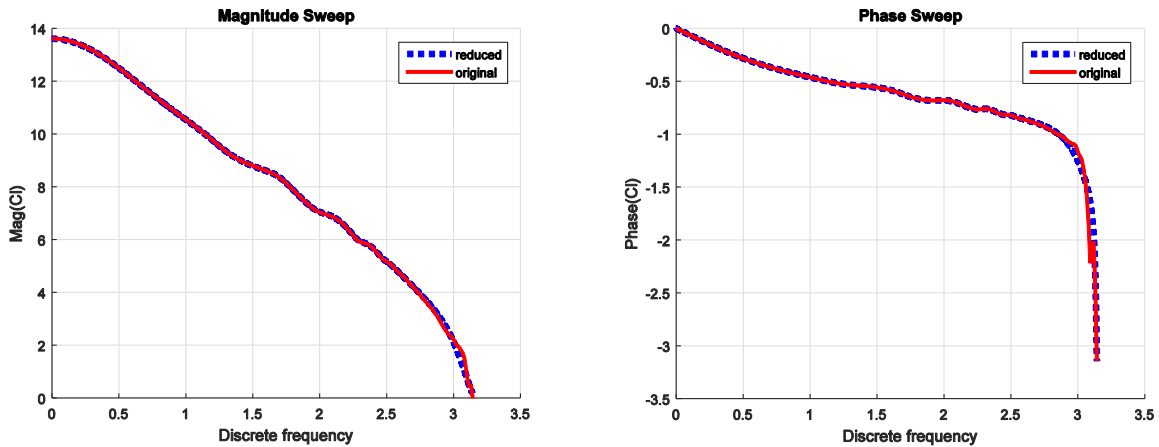


Figure 9. Magnitude & phase plot of ROM for case 1 with restarting and SS correction applied

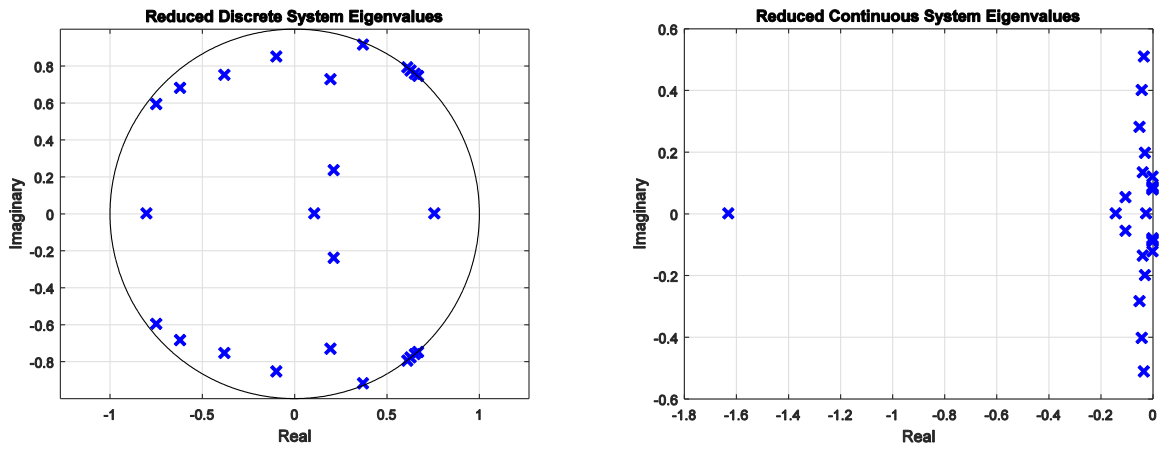


Figure 10. Eigenvalues of stable ROM for case 1

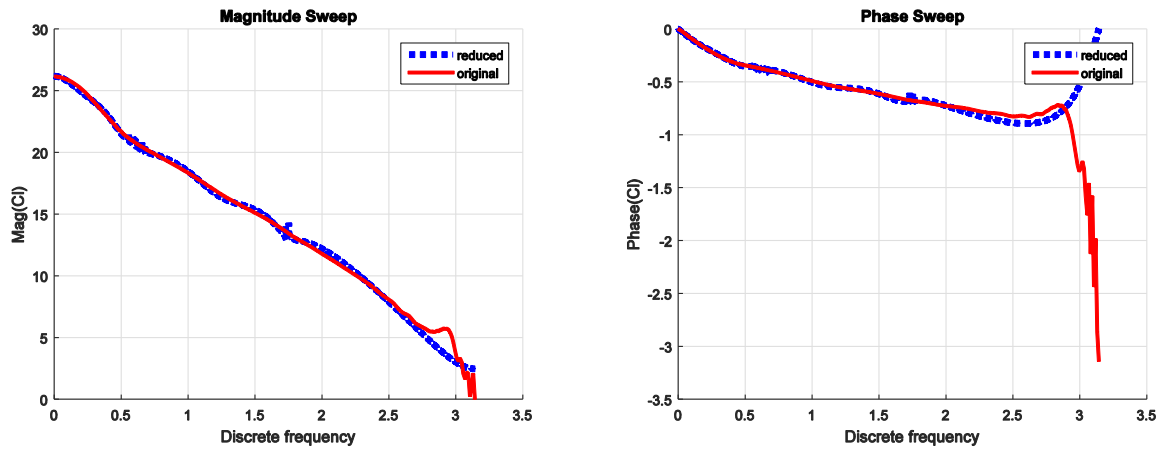


Figure 11. Magnitude & phase plot of ROM for case 2 with restarting and SS correction applied

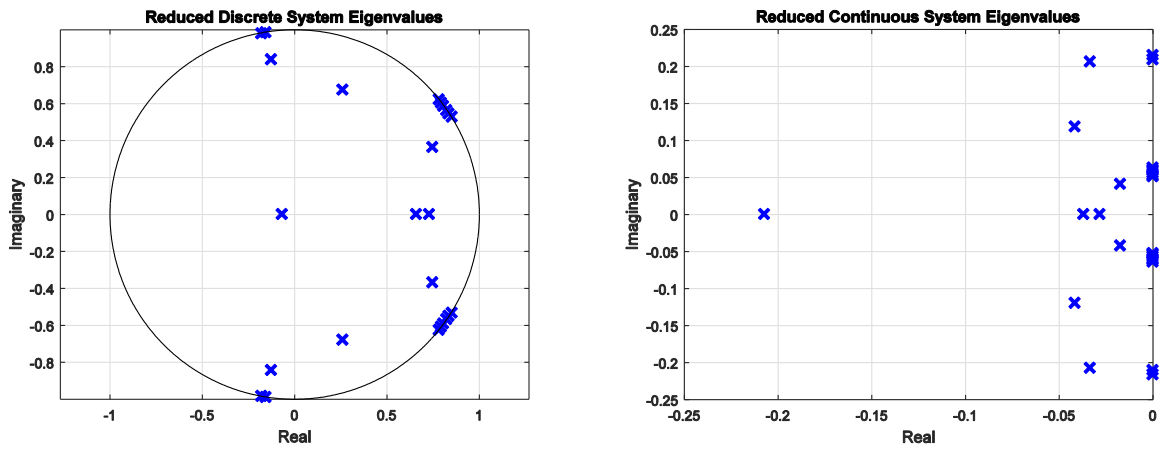


Figure 12. Eigenvalues of stable ROM for case 2

It can be seen that the restarting effectively makes the reduced system stable. Restarting can be applied as a wrapper to any existing ROM generation code to create a stable reduced model. It is noted that the ROM generated for case 2

does not fully match the original system at high frequencies, even after applying restarting. The full order system exhibits a highly oscillatory response at very high frequencies, which can be reduced by using a finer mesh around the aerofoil. Nevertheless, it was found that increasing the size of the reduced model to 55 creates a more accurate ROM which exhibits better behaviour at high frequencies, as depicted in the figures below. A total number of 18 restarts were performed to stabilise the ROM.

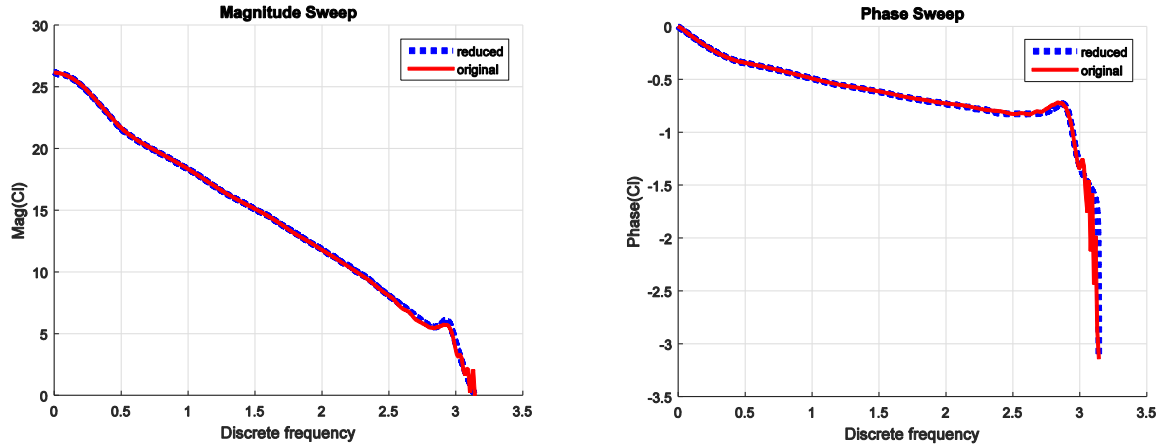


Figure 13. Magnitude & phase plot of ROM for case 2 with restarting and SS correction applied. Reduced system size = 55

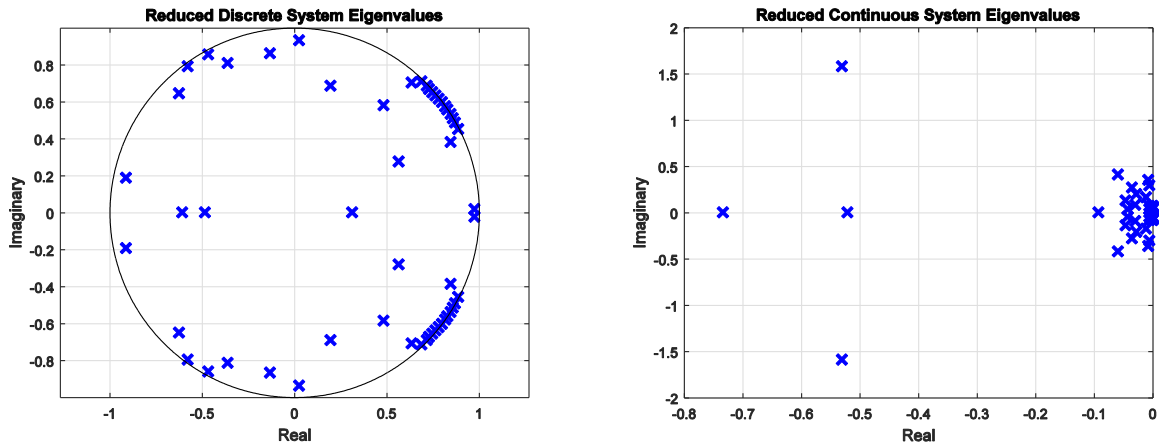


Figure 14. Eigenvalues of stable ROM for case 2. Reduced system size = 55

The ROMs were tested by prescribing a series of 1-cosine gusts and comparing the output from the ROM to that from the full order CFD simulations. The gusts prescribed are listed in the table below, and the lift coefficient plots are also given:

Table 2. 1-Cosine gust cases

Gust	Gust length	Effective change in angle of attack due to gust	Gust applied to aerofoil-Mach case
A	10 chords	5 degrees	1
B	25 chords	2 degrees	2

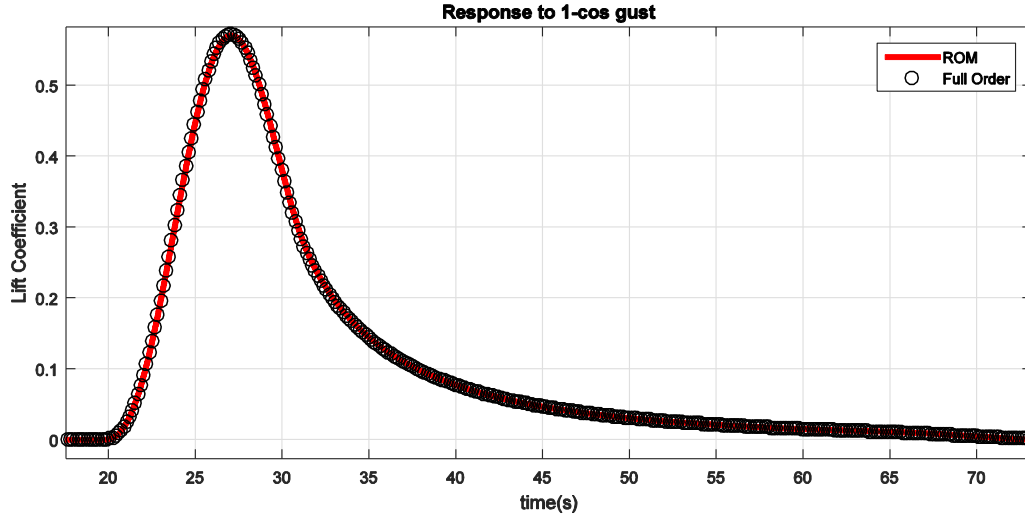


Figure 15. 1-Cosine gust “A” prescribed to ROM case 1. Reduced system size = 25.

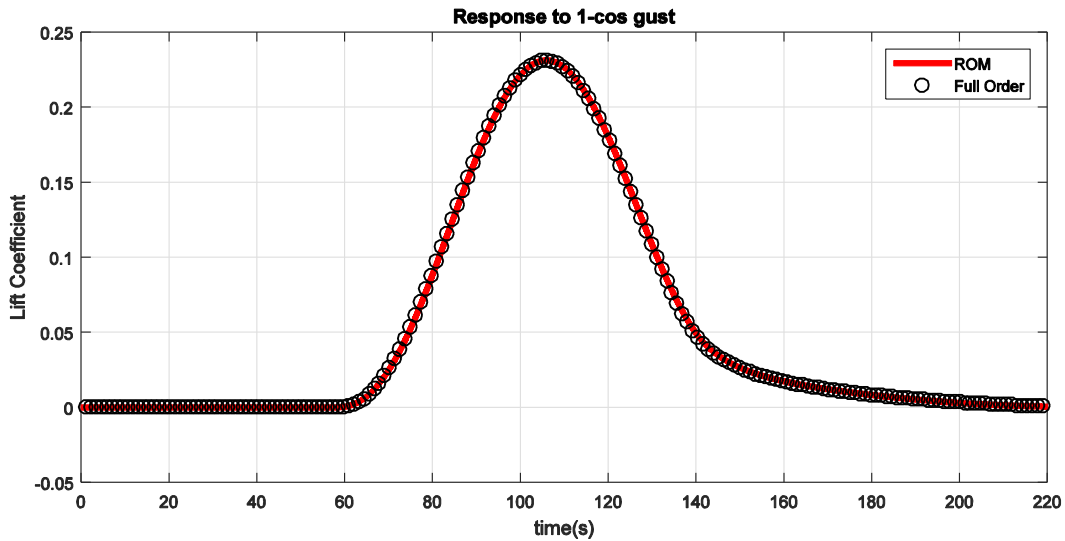


Figure 16. 1-Cosine gust “B” prescribed to ROM case 2. Reduced system size = 55.

For case 2 the ROM size was set to 55, as per Figure 14. It can be seen that for both ROMs very good agreement is achieved between the output from the ROM and the output from the full order simulation.

IV. Conclusion

It has been shown that restarting can be used to stabilise unstable ROMs. Restarting works by removing the unwanted unstable eigenvalues from the reduced system by applying shifts to the state matrices. A further method has also been introduced for correcting the steady state gain of the ROM to match the full order system. This is done by solving a constrained least squares problem to obtain the reduced system matrices.

The ROMs have been tested by comparing their output to a 1-cosine gust excitation with the output from the full order system. It has been shown that the stable ROMs created give outputs which agree very well with those from the full order system.

Future work will be aimed at creating nonlinear ROMs for flow cases with shockwaves and shock induced separation.

Acknowledgments

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